Probability theory and Linear Algebra

Question 1: Some probability basics

(a) How many different σ-algebras is it possible to define on the set $\Omega = \{A, B, C\}$? Specifically write down every possibility.

(b) Prove that the pdf of the Binomial distribution indeed qualifies as a probability function. *Hint*: You may use the Binomial theorem, which states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \,. \tag{1}$$

(c) Consider the following function

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 1, & x \geq d \\ \frac{1}{3}x^3 + \frac{1}{6}x + \frac{1}{2}, & -d \leq x \leq d \\ 0, & x \leq -d \end{cases}$$

for some $d \in \mathbb{R}_{>0}$.

(i) Determine a value for d so that F is a CDF.

(ii) Write down the corresponding probability (density) function and calculate $P(X \in A)$ for a random variable X with CDF F and

$$A = \big\{ \{x \in \mathbb{R} \mid x > 0\} \cup \{x \in \mathbb{R} \mid x \leq -d\} \big\}^C .$$

(d) Consider a test to detect a disease that 0.1% of the population have. The test is 99% effective in detecting an infected person. However, the test gives a false positive result in 0.5% of cases. If a person tests positive for the disease what is the probability that they actually have it?

Question 2: Matrix rank and linear independence

(a) Calculate the rank of the following matrix:

$$\begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

(b) Are the following vectors linearly independent?

$$v_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix} ; \; v_2 = \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \; v_3 = \begin{pmatrix} 12 \\ 1 \\ 2 \\ 4 \end{pmatrix} ; \; v_4 = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 4 \end{pmatrix} ; \; v_5 = \begin{pmatrix} 9 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

If they are not, find the largest number of linearly independent vectors among them.

(c) Prove that if a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}$ is not square, then either the row vectors or the column vectors are linearly dependent.

Question 3: Matrixdecomposition

Consider the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{und} \quad \boldsymbol{D} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 26 \end{pmatrix}.$$

You may assume it to be known that the matrix $\boldsymbol{D}$ is positive definite.

a) Determine the definitiveness of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

b) Decompose the matrix $\boldsymbol{A}$ using Eigendecomposition.

c) Determine the entries of $\boldsymbol{L}$ in the Cholesky-decomposition $\boldsymbol{D} = \boldsymbol{L}\boldsymbol{L}^T$ of $\boldsymbol{D}$.